ON THE CYCLICITY OF THE PERIOD ANNULUS OF QUADRATIC HAMILTONIAN TRIANGLE VECTOR FIELD

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ABSTRACT. This paper is concerned with the cyclicity of the period annulus of quadratic Hamiltonian triangle vector field under quadratic perturbations. This problem has been studied by Iliev (J. Differential Equations **128**(1996)), based on the displacement function obtained by Żoładek (J. Differential Equations **109**(1994)). Recently, P. Mardešić etc. (J. Dynamical and Control Systems **17**(2011)) studied unfoldings of the Hamiltonian triangle within quadratic vector fields. It turned out that the displacement function is not precise of the form given by Żoładek. Using the corrected form of the displacement function obtained by P. Mardešić etc, it is proved in this paper that the cyclicity of the period annulus under quadratic perturbations is equal to three.

1. INTRODUCTION AND THE MAIN RESULT

In this paper we study the bifurcations of limit cycles in a class of planar quadratic integrable systems under small quadratic perturbations. Taking a complex coordinate $z = x + iy$ and using the terminology from [22], the list of quadratic centers at $z = 0$ looks as follows:

$$\begin{array}{ll}
\dot{z} = -iz - z^2 + 2|z|^2 + (b+ic)\bar{z}^2, & \text{Hamiltonian}\,(Q_3^H),\\
\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2, & \text{reversible}\,(Q_3^R),\\
\dot{z} = -iz + 4z^2 + 2|z|^2 + (b+ic)\bar{z}^2, & |b+ic| = 2, \text{ codimension four}\,(Q_4),\\
\dot{z} = -iz + z^2 + (b+ic)\bar{z}^2, & \text{generalized Lotka} - \text{Volterra}\,(Q_3^{LV}),\\
\dot{z} = -iz + \bar{z}^2, & \text{Hamiltonian triangle},
\end{array}$$

where a, b, and c stand for arbitrary real constants.

Recently, the authors of the paper [17] studied unfoldings of quadratic Hamiltonian triangle vector field within the quadratic vector fields. These unfolding have been studied by Żoładek [21, 22], Iliev [9, 10] and others. It turned out that the displacement function is not precise of the form given by Żoładek.

Based on the displacement function obtained by Żoładek [22], Iliev [9] studied the cyclicity of the period annulus of quadratic Hamiltonian triangle vector field under quadratic perturbations. Due to the error in Żoładek's paper, the problem of the cyclicity of the period annulus of quadratic Hamiltonian triangle vector filed is still open. As usual, we use the notion of *cyclicity* for the total number of

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limit cycles which can emerge from a configuration of trajectories (center, period annulus, a singular loop) under a perturbation.

To state the main results of the paper [17], consider the quadratic system

$$X:\quad \begin{cases} \dot{x} &=& x[\beta-\beta x-(\beta+1)y], \\ \dot{y} &=& y[-\alpha+(\alpha+1)x+\alpha y], \end{cases} \tag{1}$$

which has a first integral $x^{\alpha}y^{\beta}(1-x-y)$ with the integrating factor $x^{\alpha-1}y^{\beta-1}$. System (1) has a center at $p_{\alpha\beta}(\alpha/(\alpha+\beta+1), \beta/(\alpha+\beta+1))$. The vector field X is Darboux integrable. If $\alpha=\beta$, then $X\in Q_3^R$. If $\alpha=\beta=1$, then X is a Hamiltonian triangle system.

According to [22], the following system gives the general unfolding of the Hamiltonian triangle system within quadratic systems:

$$X_{\varepsilon}:\quad \begin{cases} \dot{x} &=& x[\beta-\beta x-(\beta+1)y]+\varepsilon_0x^2+\varepsilon_1y^2, \\ \dot{y} &=& y[-\alpha+(\alpha+1)x+\alpha y]+\varepsilon_0y^2+\varepsilon_2x^2, \end{cases} \tag{2}$$

where $\varepsilon_0, \varepsilon_1$ and ε_2 are small parameters.

It is well known that the displacement function δ is the difference between the first return Poincaré map and the identity. We study the displacement function only in a compact domain within the basin of the center $p_{\alpha\beta}$ away from the separatrices bounding the basin and the center itself. Set

$$\varepsilon_3=\alpha-1,\ \varepsilon_4=\beta-1,\ \varepsilon=(\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4).$$

The main result of the paper [17] is the following:

Theorem 1. [17] *Let*

$$H_{\varepsilon_3,\varepsilon_4}=x^{1+\varepsilon_3}y^{1+\varepsilon_4}(1-x-y).$$

The displacement function $\delta(h,\varepsilon)$ of system (2), for ε close to 0 is of the form

$$\delta(h,\varepsilon)=\mu_1[J_1+\cdots]+\mu_2[J_2+\cdots]+\mu_3[J_3+\cdots]+\mu_4[J_4+\cdots], \tag{3}$$

where

$$\begin{aligned} \mu_1 &= -\varepsilon_0, \\ \mu_2 &= -(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4)/2, \\ \mu_3 &= (\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4)(\varepsilon_3-\varepsilon_4)/2, \\ \mu_4 &= (\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4)(\varepsilon_1+\varepsilon_2)/6, \end{aligned}$$

and

$$
\begin{aligned}
J_1 &= \oint_{h=H_{0,0}} y^2 dx - x^2 dy = -2\int\int_{h\le H_{0,0}} (x+y)dx\wedge dy,\\
J_2 &= \int\int_{h\le H_{0,0}} (xy)^{-1}(x^3+y^3)dx\wedge dy,\\
J_3 &= \int\int_{h\le H_{0,0}} (xy)^{-1}[(x-y)(x+y)^2\ln(x/y)]dx\wedge dy\\
&\quad +\int\int_{h\le H_{0,0}} (xy)^{-1}(x+y)(x^2+xy+y^2)dx\wedge dy,\\
J_4 &= \frac{1}{3h}\oint_{h=H_{0,0}} \frac{x^3y^3}{y-x}(dx+dy)\\
&= \frac{1}{3h}\int\int_{h\le H_{0,0}} \frac{x^2y^2(3y^2-4xy+3x^2)}{(x-y)^2}dx\wedge dy,
\end{aligned}
$$

and the three dots $\cdots$ *denote analytic functions in* (h,ε)*, which tend to zero as* $\varepsilon\to 0$*.*

As mentioned in [17], the function $J_4(h)$ does not vanish at the value of the center $p_{\alpha\beta}$, whereas Żoładek's does. This might seem surprising. An explanation is given in [17].

In this paper we study the cyclicity of the period annulus of quadratic Hamiltonian triangle vector field. The main result of the present paper is the following.

Theorem 2. *For small* ε*, the maximum number of limit cycles in* (2) *which emerge from the period annulus of system* (1) *is equal to three. The upper bound is sharp.*

Theorem 2 is proved by counting the number of zeros of the principal part $J(h)$ of the displacement function $\delta(\varepsilon,h)$. To do it, we show that $J'(h)$ has at most three zeros. Note that $J(h)$ does not vanish identically at the value of the center $p_{\alpha\beta}$, whereas most of the principal part of the displacement function for other integrable systems do. Hence we need a careful analysis to get the exact upper bound of the number of zeros of $J(h)$.

Let us list here some results concerning the cyclicity of the period annulus for quadratic centers. The cyclicity of the period annulus for quadratic Hamiltonian Q_3^H was completely solved by several authors, see [1, 2, 4, 8, 13, 19] and references therein. The generalized Lotka-Volterra Q_3^{LV} was studied by Żoładek in [22]. Some results concerned with certain specific cases of Q_3^R can be found in [3, 15, 16], etc. The cyclicity of the period annulus for Q_4 was investigated in [5, 18]. For more results on this problem, we refer the readers to the survey paper [12].

The rest of this paper is organized as follows. In Section 2 we give alternative expressions for the integrals $J(h)$ and some preliminary results. Section 3 is devoted to prove the monotonicity and convexity of the ratio of two integrals $w(h)=I_2'(h)/I_0'(h)$. The main result is proved in Section 4.

2. Preliminary results

To use the results obtained in [9], we begin our investigation by a technical lemma giving alternative expression for the integrals $J_k(h), k = 1, 2, 3, 4$.

Lemma 3. *Let*

$$I_i(h) = \oint_{\Gamma_h} x^i y dx,\ \ i = 1, 2, \cdots, I_*(h) = \oint_{\Gamma_h} y(x-1)\ln x dx,$$

where Γ_h is the closed component of the level set

$$H(x,y) = x(y^2 - (x-3)^2) = h,\ \ h \in (-4, 0). \tag{4}$$

Suppose that the oval Γ_h has the positive (counterclockwise) orientation. Then the following relations between the integral J_k and I_k hold:

$$\begin{aligned} J_1 &= \frac{2}{27} I_0, \\ J_2 &= \frac{2}{27h}((h+18)I_0 - 12I_2), \\ J_3 &= \frac{1}{18h}((h-12)I_0 + 24I_2 - 36I_*), \\ J_4 &= \frac{1}{10368h}((19h+702)I_0 - 324I_2) \\ &\quad + \frac{1}{20736h}((-3888 - 324h + 7h^2)I_0' + 216(6+h)I_2'). \end{aligned}$$

Proof. First of all we list several relations between $I_k(h)$, which will be used in the proof of this lemma. It has been proved that the following two identities holds in [9]:

$$I_0(h) \equiv I_1(h),\ \ (2k+6)I_{k+1} = 6(2k+3)I_k - 18kI_{k-1} - (2k-3)hI_{k-2}. \tag{5}$$

One obtain from (4) that

$$dH = 2xydy + (-9 + 12x - 3x^2 + y^2)dx.$$

Multiplying both sides of the above equation by $x^i y^{j-2}$ and integrating over Γ_h yields

$$0 = 2\oint_{\Gamma_h} x^{i+1} y^{j-1} dy - 9I_{i,j-2} + 12I_{i+1,j-2} - 3I_{i+2,j-2} + I_{ij},$$

where

$$I_{ij} = I_{ij}(h) = \oint_{\Gamma_h} x^i y^j dx.$$

On the other hand,

$$\oint_{\Gamma_h} x^{i+1} y^{j-1} dy = -\frac{i+1}{j} I_{ij}.$$

Therefore,

$$\frac{j-2i-2}{j} I_{ij} = 9I_{i,j-2} - 12I_{i+1,j-2} + 3I_{i+2,j-2}. \tag{6}$$

It follows from (4) that

$$d\left(\ln\frac{3-x-y}{3-x+y}\right) = \frac{2xy}{H}dx - \frac{2x(x-3)}{H}dy, \tag{7}$$

and

$$I_i'(h) = \oint_{\Gamma_h} x^i \frac{\partial y}{\partial h}dx = \oint_{\Gamma_h} \frac{x^{i-1}}{2y}dx. \tag{8}$$

Perform a change of the variables which transforms the Hamiltonian $H_{0,0}$ into H, defined in (4). This is given by $(x, y, H_{0,0}, h) \to (x_1, y_1, H, h_1)$, where

$$x = \frac{1}{6}(3 - x_1 + y_1),\ \ y = \frac{1}{6}(3 - x_1 - y_1),\ \ h_1 = -108h. \tag{9}$$

Below we calculate in detail the formulae for J_2 and J_3. The remaining relations can be obtained in a similar way. By changing the variables according to (9) and using (6), (7) and Green's formula, we get (omitting the subscript 1)

$$\begin{aligned}
J_2 &= -\frac{1}{54}\int\int_{H\le h} \frac{(-3+x)(9-6x+x^2+3y^2)}{(-3+x-y)(-3+x+y)}dx\wedge dy \\
&= -\frac{1}{18}\oint_{\Gamma_h}(x-3)y\,dx + \frac{1}{27}\oint_{\Gamma_h}(x-3)^2\ln\frac{3-x-y}{3-x+y}dx \\
&= \frac{1}{9}I_0 + \frac{1}{81}\oint_{\Gamma_h}\ln\frac{3-x-y}{3-x+y}d(27x-9x^2+x^3) \\
&= \frac{1}{9}I_0 - \frac{1}{81}\oint_{\Gamma_h}(27x-9x^2+x^3)d\ln\frac{3-x-y}{3-x+y} \\
&= \frac{1}{9}I_0 - \frac{1}{81}\oint_{\Gamma_h}(27x-9x^2+x^3)\left(\frac{2xy}{H}dx - \frac{2x(x-3)}{H}dy\right) \\
&= \frac{1}{9}I_0 - \frac{2}{81h}\left(27I_2 - 9I_3 + I_4 + \oint_{\Gamma_h} y\,d(x(x-3)(27x-9x^2+x^3))\right) \\
&= \frac{1}{9}I_0 - \frac{2}{81h}(27I_2 - 9I_3 + I_4 - 162I_1 + 162I_2 - 48I_3 + 5I_4) \\
&= \frac{2}{27h}((h+18)I_0 - 12I_2).
\end{aligned}$$

Rewrite J_3 as the form

$$J_3 = J_{31} + J_{32},$$

where

$$J_{31} = \int\int_{h\le H_{0,0}} (xy)^{-1}(x-y)(x+y)\left[(x+y)\ln\frac{x}{y} - 2(x-y)\right]dx\wedge dy,$$

and

$$\begin{aligned} J_{32} &= \int\int_{h\le H_{0,0}} \frac{2(x-y)^2(x+y)+(x+y)(x^2+xy+y^2)}{xy} dx\wedge dy \\ &= 3\int\int_{h\le H_{0,0}} (xy)^{-1}(x^3+y^3) dx\wedge dy \\ &= 3J_2. \end{aligned}$$

Note that J_{31} is the integral $J_3(h)$ defined in the papers [22, 9], see §2.2 in [22] or Lemma in [9]. Taking $k=1$ into (5), one gets the expression of I_{-1}. It follows from Lemma, Lemma 1 and the appendix of [9] that

$$J_{31} = \frac{1}{2}I_{-1} + \left(\frac{4}{3h}-\frac{1}{6}\right)I_0 - \frac{2}{h}I_* = \frac{1}{6h}((-28-h)I_0 + 24I_2 - 12I_*)).$$

The formula for $J_3(h)$ follows from the expression of J_{31} and J_{32}. □

The following lemma is crucial for our analysis.

Lemma 4. [9] *The vector-function* $\mathrm{col}(I_*, I_2, I_0)$ *satisfies the following system*

$$\begin{pmatrix} I_* \\ I_2 \\ I_0 \end{pmatrix} = \begin{pmatrix} h & -2 & h+6 \\ 0 & \frac{3}{4}(h-6) & \frac{3}{2}(h+9) \\ 0 & -3 & \frac{3}{2}(h+6) \end{pmatrix} \begin{pmatrix} I_*' \\ I_2' \\ I_0' \end{pmatrix}. \tag{10}$$

Lemma 5. $I_0'(h) < 0$ *for* $h \in (-4, 0)$.

Proof. Let $x_1(h)$ and $x_2(h)$ be the intersection point of Γ_h and x-axis with $x_1(h) < x_2(h)$. Note that Γ_h is located in the right half-plane and has the positive (counterclockwise) orientation. It follows from (8) and (4) that

$$\begin{aligned} I_0'(h) &= \oint_{\Gamma_h} \frac{x^{-1}}{2y} dx \\ &= \int_{x_2(h)}^{x_1(h)} \frac{x^{-1}}{2\sqrt{h/x+(x-3)^2}} dx + \int_{x_1(h)}^{x_2(h)} \frac{x^{-1}}{2(-\sqrt{h/x+(x-3)^2})} dx \\ &= -\int_{x_1(h)}^{x_2(h)} \frac{x^{-1}}{\sqrt{h/x+(x-3)^2}} dx < 0. \end{aligned}$$

□

Define

$$w(h) = \frac{I_2'(h)}{I_0'(h)}. \tag{11}$$

Let $J(h)$ be the principal part of the displacement function $\delta(\varepsilon, h)$, i.e.

$$J(h) = \mu_1 J_1 + \mu_2 J_2 + \mu_3 J_3 + \mu_4 J_4. \tag{12}$$

The function $J(h)$ can be expressed in a more convenient form by using Picar-Fuchs system (10). Therefore, we reformulate Theorem 1 as follows.

Proposition 6. *For small ε, the number of limit cycles in* (2) *that emerge from the period annulus is equal to the number of zeros in the interval* $(-4,0)$ *of the function $J(h)$, where*

$$\begin{aligned} J(h) \; &= \; \frac{2(3\lambda-12\sigma-10\gamma+72\kappa)+(\lambda-4\sigma+2\gamma-8\kappa)h}{5184}I_0' \\ &\quad + \frac{-\lambda+4\sigma-2\gamma-24\kappa}{2592}I_2' + \frac{\lambda+6\gamma}{1296}I_*', \end{aligned} \tag{13}$$

with

$$\begin{aligned} &\lambda = -1296\mu_2 - 648\mu_3, \quad \sigma = -144\mu_1 - 468\mu_2 - 270\mu_3 - 6\mu_4, \\ &\gamma = 216\mu_2 - 324\mu_3, \qquad \kappa = 54\mu_2 - 81\mu_3 + \mu_4. \end{aligned} \tag{14}$$

Moreover,

$$J'(h) = \frac{f(h)I_0'(h)}{7776h(4+h)}, \tag{15}$$

where

$$f(h) = -16\lambda - 16\sigma h + (\lambda-4\sigma+2\gamma-8\kappa)h^2 - 32(\gamma+\kappa h)w. \tag{16}$$

The corresponding integrals and function are given in Lemma 3 and (11).

Proof. We will prove in Corollary 23 (see Section 4 below) that $J(h)\equiv 0$ if and only if $\mu_1=\mu_2=\mu_3=\mu_4=0$. Therefore, it follows from (3) that $\delta(\varepsilon,h)\equiv 0$ if and only if $\mu_1=\mu_2=\mu_3=\mu_4=0$. This implies that the number of limit cycles in (2) that emerge from the period annulus is equal to the number of zeros in the interval $(-4,0)$ of the function $J(h)$, defined in (12), where $J_k(h),\ k=1,2,3,4$, are given in Lemma 3.

The expression (13) follows from (12) and (10). To calculate $J'(h)$, we firstly derive several formulae. Differentiating both sides of system (10), we have

$$\begin{pmatrix} h & -2 & h+6 \\ 0 & \frac{3}{4}(h-6) & \frac{3}{2}(h+9) \\ 0 & -3 & \frac{3}{2}(h+6) \end{pmatrix} \begin{pmatrix} I_*'' \\ I_2'' \\ I_0'' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{1}{4} & -\frac{3}{2} \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} I_2' \\ I_0' \end{pmatrix},$$

which implies

$$3h(h+4)\begin{pmatrix} I_2'' \\ I_0'' \end{pmatrix} = \begin{pmatrix} h+6 & -2(9+2h) \\ 2 & -6-h \end{pmatrix}\begin{pmatrix} I_2' \\ I_0' \end{pmatrix}, \tag{17}$$

and

$$I_*'' = -\frac{2}{3h}I_0'. \tag{18}$$

Differentiating both sides of system (13) and then taking (17), (18) into it, one obtains (15). □

Remark 7. *The changes* (14) *transform the parameters* $\mu_1, \mu_2, \mu_3, \mu_4$ *into* $\lambda, \sigma, \gamma, \kappa$. *In the rest of this paper, we also take* $\lambda, \sigma, \gamma, \kappa$ *as the independent parameters, instead of* μ_i, $i = 1, 2, 3, 4$. *The function* $f(h)$ *is expressed in a more convenient form by the new parameters.*

Without loss of generality suppose $\kappa = 1$ *if* $\kappa \neq 0$. *That is to say,* $\kappa \in \{0, 1\}$.

In the end of this section we give the asymptotic expansions of the Abelian integrals $I_*(h), I_2(h)$ and $I_0(h)$ near the endpoints of $(-4, 0)$.

Lemma 8. [9] *The following expansions hold near* $h = -4$, *with* $I_0'(-4) < 0$:

$$\begin{aligned}
I_* &= I_0'(-4)\left(\frac{(h+4)^2}{12} + \frac{11(h+4)^3}{1296} + \frac{109(h+4)^4}{93312} + \cdots\right), \\
I_2 &= I_0'(-4)\left(h + 4 + \frac{(h+4)^2}{9} + \frac{17(h+4)^3}{1944} + \frac{455(h+4)^4}{419904} + \cdots\right), \\
I_0 &= I_0'(-4)\left(h + 4 + \frac{(h+4)^2}{36} + \frac{5(h+4)^3}{1944} + \frac{35(h+4)^4}{104976} + \cdots\right).
\end{aligned}$$

Lemma 9. *The following expansions hold near* $h = 0$:

$$\begin{aligned}
I_* &= -6 - \frac{1}{6} h \ln^2 |h| + \cdots, \\
I_2 &= -\frac{27}{2} + \frac{3}{2} h \ln |h| + 3(c+1)h + \cdots, \\
I_0 &= -9 + \frac{1}{2} h \ln |h| + ch + \cdots,
\end{aligned}$$

where c *is a real constant.*

Proof. The first two coefficients of every integral have been obtained in [9]. It is well known (see [14], or Appendix of [20]) that the asymptotic expansion of I_2 and I_0 near $h = 0$ have the form $\varphi_0(h) + \varphi_1(h) \ln |h|$, where $\varphi_0(h)$ and $\varphi_1(h)$ are analytic at $h = 0$. Let

$$I_2 = -\frac{27}{2} + \frac{3}{2} h \ln |h| + c_2 h + \cdots, \quad I_0 = -9 + \frac{1}{2} h \ln |h| + ch + \cdots.$$

Taking the above expansions into the last two equations of system (10), we get $c_2 = 3(c + 1)$. □

Using the above two lemmas, we get the following expansions for $w(h)$.

Corollary 10. *The following expansion holds near* $h = -4$:

$$w(h) = 1 + \frac{1}{6}(h+4) + \frac{1}{108}(h+4)^2 + \frac{7}{5832}(h+4)^3 + \cdots. \tag{19}$$

The following expansion holds near $h = 0$:

$$w(h) = 3 + \frac{6}{\ln |h|} + \cdots. \tag{20}$$

3. The geometric properties of the ratio $w(h)$

In this section we study the monotonicity, convexity, etc. for the ratio $w(h)$.

Lemma 11. *Then the ratio $w(h)$, defined in (11), satisfies the following equation*

$$3h(h+4)w' = -2w^2 + 2(h+6)w - 2(2h+9). \tag{21}$$

Proof. The equation follows from system (17). □

Proposition 12. $w(-4) = 1,\ w(0) = 3$, *and* $w'(h) > 0$ *for* $h \in (-4, 0)$.

Proof. The equalities $w(-4) = 1$ and $w(0) = 3$ are obtained from (19) and (20). Since the right hand of equation (21) is a quadratic polynomial in w and $(2(h+6))^2 - 4(-2)(-2(2h+9)) = 4h(h+4) < 0$ for $h \in (-4, 0)$, we have $-2w^2 + 2(h+6)w - 2(2h+9) < 0$ in the interval $(-4, 0)$. It follows from (21) that $w'(h) > 0$ for $h \in (-4, 0)$. □

Assume that $\varphi(x_1, x_2)$ and $\phi(x_1, x_2)$ are two polynomials in x_1, x_2. Eliminating the variable x_i from the equations $\varphi(x_1, x_2) = \phi(x_1, x_2) = 0$, we get the resultant of φ and ϕ, denoted by Resultant(φ, ϕ, x_i).

Proposition 13. *For* $h \in (-4, 0)$,$w''(h) > 0,\ w'''(h) > 0$.

Proof. Rewrite (21) as the form

$$w'(h) = \frac{-2w^2 + 2(h+6)w - 2(2h+9)}{3h(h+4)}. \tag{22}$$

By abuse of notation we denote w' the right hand of (22), which is a function of two variables h and w. Then

$$w''(h) = \frac{\partial w'}{\partial h} + \frac{\partial w'}{\partial w} w' = -\frac{2(6+h-2w)(-2h-6w+hw+2w^2)}{9h^2(4+h)^2}. \tag{23}$$

Similarly let w'' be the function of the right hand of (23). Using (22) and (23) yields

$$w'''(h) = \frac{\partial w''}{\partial h} + \frac{\partial w''}{\partial w} w' = \frac{4\zeta(h,w)}{27h^3(4+h)^3}, \tag{24}$$

where

$$\begin{aligned} \zeta(h,w) &= -324 - 108h - 37h^2 - 4h^3 + 2(108 - 42h + 4h^2 + h^3)w \\ &\quad + (-144 + 76h + h^2)w^2 - 12(h-6)w^3 - 12w^4. \end{aligned}$$

Let

$$\Theta = \{(h,w) | -4 \le h \le 0,\ 1 \le w \le 3\}.$$

It follows from Proposition 12 that $(h, w(h)) \in \Theta$ for $h \in [-4, 0]$.

The maximum and minimum for $\zeta(h,w)$ in Θ necessarily occur either on the boundary $\partial\Theta$ of the set Θ, or the points inside Θ whose coordinates satisfy equations

$$\zeta_h = \frac{\partial \zeta}{\partial h} = 0,\ \zeta_w = \frac{\partial \zeta}{\partial w} = 0. \tag{25}$$

Note

$$\zeta(-4,w) = -12(w-1)^2(19-8w+w^2), \quad \zeta(0,w) = -12(w-3)^2(w^2+3),$$

$$\zeta(h,1) = -2(4+h)^2(6+h), \qquad \zeta(h,3) = 2h^2(h-2).$$

This implies that $\zeta(h,w) < 0$ in $\partial\Theta \backslash \{(-4,1),(0,3)\}$ and $\zeta(-4,1) = \zeta(0,3) = 0$.

On the other hand, we will show that $\zeta(h,w)$ has a unique negative minimum value inside Θ. To this end, we eliminate the variable h or w from the equations (25) to get

$$\begin{aligned}\text{Resultant}(\zeta_h, \zeta_w, h) &= -1024(w-3)^2(w-2)(w-1)^2\chi_1(w),\\ \text{Resultant}(\zeta_h, \zeta_w, w) &= -6144h^2(h+2)(h+4)^2\chi_2(h).\end{aligned}$$

where

$$\begin{aligned}\chi_1(w) &= -21975 + 14660w - 1841w^2 - 912w^3 + 114w^4,\\ \chi_2(h) &= -170856 - 4036h - 401h^2 + 304h^3 + 38h^4.\end{aligned}$$

It is easy to obtain

$$\chi_1'(w) = 2(w-2)(228w^2 - 912w - 3665), \quad \chi_2'(h) = 2(h+2)(76h^2 + 304h - 1009).$$

It is readily verified that $\chi_1'(w)$ (resp. $\chi_2'(h)$) has a unique zero $w = 2$ (resp. $h = -2$) in the interval $(1,3)$ (resp. $(-4,0)$). Thus, $\chi_1(w)$ (resp. $\chi_2(h)$) has a maximum or minimum value at $w = 2$ (resp. $h = -2$). By direct computation we have $\chi_1(1) < 0$, $\chi_1(2) < 0$, $\chi_1(3) < 0$ (resp. $\chi_2(-4) < 0$, $\chi_2(-2) < 0$, $\chi_2(0) < 0$), which implies $\chi_1(w) < 0$ (resp. $\chi_2(h) < 0$) for $w \in (1,3)$ (resp. $h \in (-4,0)$. Therefore, the equations (25) has a unique solution $(-2,2)$ in $\Theta\backslash\partial\Theta$. This yields that $\zeta(h,w)$ has a minimum value $\zeta(-2,2) = -16$. Consequently $\zeta(h,w) \le 0$ for $(h,w) \in \Theta$.

Finally we prove $\zeta(h,w) < 0$ for $(h,w) \in \Theta\backslash\partial\Theta$. We have known $\zeta(h,w) \le 0$ for $(h,w) \in \Theta$. If $(h^*,w^*) \in \Theta\backslash\partial\Theta$ and $\zeta(h^*,w^*) = 0$, then $\zeta(h,w)$ has a maximum value at (h^*,w^*) satisfying $\zeta_h(h^*,w^*) = \zeta_w(h^*,w^*) = 0$. By the above discussion we know $(h^*,w^*) = (-2,2)$, which yields a contradiction.

It follows from $\zeta(h,w) < 0$ and (24) that $w'''(h) > 0$ for $h \in (-4,0)$. Hence $w''(h)$ is an increasing function in $(-4,0)$. By (19) one gets $w''(-4) = 1/54 > 0$. Consequently $w''(h) > 0$ for $h \in (-4,0)$. This completes the proof. $\square$

Corollary 14. *Let*

$$l_1(h) = 1 + \frac{1}{6}(h+4), \quad l_2(h) = 3 + \frac{h}{2}. \tag{26}$$

Then in the hw-plane, $(h, w(h)) \in \mathcal{D}$, where

$$\mathcal{D} = \{(h,w) | -4 \le h \le 0, \ l_1(h) \le w \le l_2(h)\}. \tag{27}$$

Proof. Proposition 12 and Proposition 13 show that $w(h)$ is increasing and strictly convex in $[-4,0)$. The points $S_1(-4,1)$ and $S_2(0,3)$ are the endpoints of the curve $w(h)$ in hw-plane. Note that $w = l_1(h)$ is the tangents to $w(h)$ at S_1 and $w = l_2(h)$ is the straight line passing through both S_1 and S_2. The assertion follows. $\square$

4. Proof of Theorem 2

In this section we prove Theorem 2.

Proposition 15. *If $\kappa = 0$, then $J(h)$ has at most three zeros in* $(-4, 0)$

Proof. If $\kappa = \gamma = 0$, then $f(h) = (h+4)(-4\lambda + (\lambda - 4\sigma)h)$, which implies that $f(h)$ has at most one zero in $(-4, 0)$. It follows from (15) that $J'(h)$ has at most one zero. Hence, $J(h)$ has at most two zeros in $(-4, 0)$.

If $\kappa = 0,\ \gamma \neq 0$, then $f(h) = -32\gamma w - 16\lambda - 16\sigma h + (\lambda - 4\sigma + 2\gamma)h^2$. Therefore, it follows from Proposition 13 that $f'''(h) = -32\gamma w'''(h) \neq 0$. Noting $f(-4) = 0$, this yields that $f(h)$ has at most two zeros. Consequently, $J(h)$ has at most three zeros. □

Now we consider the case $\kappa \neq 0$. To simplify the proofs, we suppose $\kappa = 1$, as mentioned in Remark 7. By (16), we get

$$f'''(h) = -96w'''(h)\rho(h), \tag{28}$$

where

$$\rho(h) = \frac{w''(h)}{w'''(h)} + \frac{1}{3}(h + \gamma). \tag{29}$$

Lemma 16. *For $h \in (-4, 0)$, $\rho'(h) < 0$.*

Proof. By abuse of notation let ρ be the right hand of equation (29). Then it follows from (29) (22), (23) and (24) that

$$\rho'(h) = \frac{\partial \rho}{\partial h} + \frac{\partial \rho}{\partial w}w' = \frac{16\Psi_1(h, w)\Psi_2(h, w)}{2187h^6(4+h)^6(w'''(h))^2}, \tag{30}$$

where

$$\begin{aligned}
\Psi_1(h, w) &= 2(h^3 + 13h^2 + 108h + 324) - (h^3 - 2h^2 + 48h + 432)w \\
&\quad -2(h^2 + 4h - 36)w^2, \\
\Psi_2(h, w) &= -2(5h^3 + 47h^2 - 324) + (5h^3 + 62h^2 - 228h - 1080)w \\
&\quad -8(h^2 - 23h - 63)w^2 - 36(h + 2)w^3.
\end{aligned}$$

In the rest of the proof let $l_1(h),\ l_2(h)$ be the linear functions given in (26) and let $\mathcal{D}$ be the set defined in (27). Recall that Corollary 14 shows $(h, w(h)) \in \mathcal{D}$.

Firstly consider the algebraic curve $\Psi_1(h, w) = 0$. Note that $\Psi_1(h, w)$ is a quadratic polynomial in w. Since $-2(h^2 + 4h - 36) > 0$, $\Psi_1(h, l_1(h)) = -2(-9 + h)(4 + h)^3/9 > 0$ and $\Psi_1(h, l_2(h)) = -h^2(4 + h)^2 < 0$ for $h \in (-4, 0)$, there is a unique branch of the algebraic curve $\Psi_1(h, w) = 0$ for $(h, w) \in \mathcal{D}\backslash\partial\mathcal{D}$, defined by

$$\widetilde{w}(h) = \frac{h^3 - 2h^2 + 48h + 432 + h(h+4)\sqrt{h^2 + 4h + 324}}{2\left(-2h^2 - 8h + 72\right)}.$$

This shows that $\widetilde{w}(-4) = 1$, $\widetilde{w}'(-4) = 1/6$, $\widetilde{w}''(-4) = 13/162$. By direct computation, one gets for $w \in (1,3)$,

$$\text{Resultant}\left(\Psi_1, \frac{\partial \Psi_1}{\partial w}, h\right) = -746496(w-3)^2(w-1)^2(69-20w+5w^2) < 0,$$

which implies that $\Psi_{1w} \neq 0$ on the algebraic curve $\Psi_1(h,w) = 0$.

Now we are going to show that the curves $w(h)$ and $\widetilde{w}(h)$ do not intersect in $\mathcal{D}\backslash\partial\mathcal{D}$. Let

$$\upsilon(h) = w(h) - \widetilde{w}(h).$$

By direct computation, $\upsilon(h) = -5(h+4)^2/162 + \cdots < 0$ as $h \to -4$, and $\upsilon(h) = 6/\ln|h| + \cdots < 0$ as $h \to 0$. Hence, $\upsilon(h)$ have at least two zeros in $(-4,0)$, counted with their multiplicities, if the curves $w(h)$ and $\widetilde{w}(h)$ intersect in $\mathcal{D}\backslash\partial\mathcal{D}$. Assume that h_1 and h_2 are two zeros of $\upsilon(h)$ with $h_1 \leq h_2$ if possible. This yields that $\upsilon'(h)$ has at least one zero h^* such that $h^* \in (h_1,h_2)$ if $h_1 \neq h_2$, or $h^* = h_1 = h_2$. That is, the following equations have at least one solution for $h \in (-4,0)$:

$$\Psi_1(h,w) = 0, \;\; w'(h) + \frac{\Psi_{1h}}{\Psi_{1w}} = \frac{\psi(h,w)}{3h(h+4)\Psi_{1w}} = 0, \tag{31}$$

where $w'(h)$ is defined in (22), and

$$\begin{aligned}\psi(h,w) \;\;=\;\; & 2(11h^4+119h^3+714h^2+2592h+3888)-(11h^4+16h^3+32h^2 \\ & +2304h+7776)w-6(3h^3+26h^2-16h-432)w^2 \\ & +8(h^2+4h-36)w^3.\end{aligned}$$

However,

$$\text{Resultant}(\psi, \Psi_1, w) = -466560h^5(h+4)^5\left(h^2+4h-36\right) < 0, \;\; h \in (-4,0).$$

Consequently, the equations (31) has no solution for $h \in (-4,0)$, which yields contradiction. Thus, the curve $w(h)$ does not intersect $\widetilde{w}(h)$ in the set $\mathcal{D}\backslash\partial\mathcal{D}$ and $w(h) < \widetilde{w}(h)$ for $h \in (-4,0)$, which means $\Psi_1(h,w(h)) \neq 0$. Since $l_1(h) < w(h) < \widetilde{w}(h)$ and $\Psi_1(h,l_1(h)) = -2(-9+h)(4+h)^3/9 > 0$, one obtain $\Psi_1(h,w(h)) > 0$ for $h \in (-4,0)$.

Secondly we consider the algebraic curve $\Psi_2(h,w) = 0$. Note that $\Psi_2(h,w)$ is a cubic polynomial of w for $h \neq -2$. If $h \neq -2$, then $\Psi_2(h,w) = 0$ has at most three branches. Since for $h \in (-4,0)$,

$$\Psi_2(h,1) = -5h(h+4)^2 > 0, \;\; \Psi_2(h,l_1(h)) = \frac{4}{9}(h-3)(h+4)^3 < 0,$$

$$\Psi_2(h,l_2(h)) = -4h^2(h+4)^2 < 0, \;\; \Psi_2(h,3) = 5h^2(h+4) > 0,$$

we conclude that the algebraic curve $\Psi_2(h,w) = 0$ has no branch in $\mathcal{D}\backslash\partial\mathcal{D}$, one branch in $1 < w < l_1(h)$ and one branch in $l_2(h) < w < 3$ respectively, provided that $h \in (-4,0)$ and $h \neq -2$. Therefore, $\Psi_2(h,w) < 0$ for $(h,w) \in \mathcal{D}\backslash\partial\mathcal{D}$, $h \neq -2$.

If $h=-2$, then $\Psi_2(-2,w)=8(44-52w+13w^2)$. Direct computation shows that $\Psi_2(-2,w)=0$ has two zeros at

$$w_1^*=\frac{2}{13}\left(13-\sqrt{26}\right)\approx 1.21554,\ \ w_2^*=\frac{2}{13}\left(13+\sqrt{26}\right)\approx 2.78446.$$

On the other hand, the straight line $h=-2$ intersects the boundary of the set $\mathcal{D}$ at two points $(-2,4/3)$ and $(-2,2)$, which implies that $4/3\le w^*\le 2$ if $(-2,w^*)\in\mathcal{D}$. This gives $(-2,w_i^*)\notin\mathcal{D}$, $i=1,2$. Hence, $\Psi_2(-2,w)<0$ for $(-2,w)\in\mathcal{D}\backslash\partial\mathcal{D}$.

Summing the above discussions, we get $\rho'(h)<0$ from (30). □

Proposition 17. *If $\kappa=1$, $\gamma\in(-\infty,-26/7]\cup[0,+\infty)$, then $J(h)$ has at most three zeros in $(-4,0)$.*

Proof. It follows from (19) and (20) that

$$\rho(-4)=\frac{1}{21}(7\gamma+26),\ \ \rho(0)=\lim_{h\to 0}\rho(h)=\frac{1}{3}\gamma. \tag{32}$$

It follows from Lemma 16 that $\rho(h)>0$ if $\gamma\in[0,+\infty)$ and $\rho(h)<0$ if $\gamma\in(-\infty,-26/7]$, $h\in(-4,0)$, which implies that $f'''(h)\neq 0$ for $h\in(-4,0)$ by (28) and Proposition 13. Hence, $f''(h)$ has at most one zero. Noting $f(-4)=0$, $f(h)$ has at most two zeros in $(-4,0)$. One obtains by using (15) that $J(h)$ has at most three zeros. □

The following lemma gives the asymptotic properties for $f(h)$ and $J(h)$.

Lemma 18. *Let $\kappa=1$. The following identities hold:*

$$\begin{aligned} &f(-4)=0,\ \ f'(-4)=-\frac{8}{3}(3\lambda-6\sigma+8\gamma-20),\\ &J(-4)=\frac{4-\gamma}{162}I_0'(-4),\ \ J'(-4)=-\frac{1}{31104}f'(-4)I_0'(-4). \end{aligned}\tag{33}$$

The following expansions hold near $h=0$:

$$J(h)=\frac{f(0)}{124416}\ln^2|h|+\cdots,\ \ f(h)=f(0)-\frac{192\gamma}{\ln|h|}+\cdots, \tag{34}$$

where

$$f(0)=-16(\lambda+6\gamma).$$

Proof. The results follows from (13), (16), (19), (20), Lemma 8 and Lemma 9. □

Proposition 19. *Suppose $\kappa=1$, $\gamma\in(-26/7,0)$ and $h\in(-4,0)$. If $f(0)\ge 0$, then $J(h)$ has at most three zeros; If $f(0)<0$, then $J(h)$ has at most two zeros.*

Proof. If $\kappa=1$, $\gamma\in(-26/7,0)$, then it follows from (32) and Lemma 16 that $\rho(h)$ has a unique zero in $(-4,0)$, which implies that $f''(h)$ has at most two zeros by (28). Hence $f'(h)$ has at most three zeros. Noting $f(-4)=0$, $f(h)$ has at most three zeros in $(-4,0)$. This yields that $J'(h)$ has at most three zeros by (15). Therefore $J(h)$ has at most four zeros in $(-4,0)$.

If $\gamma<0$, then it follows from (33) and Lemma 5 that $J(-4)<0$.

We split the proof into three cases.

Case 1. $f(0) > 0$.

It follows from (34) that $J(h) > 0$ as $h \to 0$. The number of zeros of $J(h)$ is odd. Hence, $J(h)$ has at most three zero in $(-4, 0)$.

Case 2. $f(0) = 0$.

Since $f''(h)$ has at most two zeros and $f(-4) = f(0) = 0$, $f(h)$ has at most two zeros in $(-4, 0)$. It follows from (15) that $J'(h)$ has at most two zeros. This implies that $J(h)$ has at most three zeros in $(-4, 0)$.

Case 3. $f(0) < 0$.

By (34) $J(h) < 0$ as $h \to 0$. Noting $J(-4) < 0$, the number of zeros of $J(h)$ is even.

If $f'(-4) < 0$, then the number of zeros of $f(h)$ is even. Therefore $f(h)$ has at most two zeros in $(-4, 0)$. This means that $J(h)$ has at most three zeros. Since the number of zeros of $J(h)$ is even, $J(h)$ has at most two zeros.

If $f'(-4) = 0$, then $f'(h)$ has at most two zeros. Since $f(-4) = 0$, $f(h)$ has at most two zeros. Consequently, $J(h)$ has at most three zeros. Noting that the number of zeros of $J(h)$ is even, $J(h)$ has at most two zeros.

It follows from Lemma 18 that $f'(h) = 192\gamma/(h \ln^2|h|) + \cdots > 0$ as $h \to 0$. If $f'(-4) > 0$, then the number of zeros of $f'(h)$ is even. Since $f'(h)$ has at most three zeros, $f'(h)$ has at most two zeros in $(-4, 0)$, which shows that $f(h)$ has at most two zeros in $(-4, 0)$. Therefore it follows from (15) that $J'(h)$ has at most two zeros, and hence $J(h)$ has at most three zeros. Since we have shown that the number of zeros of $J(h)$ is an even number, $J(h)$ has at most two zeros in $(-4, 0)$.

The proof is finished. $\square$

By the above discussions we know that $J(h)$ has at most three zeros in $(-4, 0)$. In what follows we are going to show that there exist the parameters $\mu_i, i = 1, 2, 3, 4$, such that $J(h)$ has exactly three zeros in $(-4, 0)$. To do this we introduce the following definitions (see for instance [6, 11]).

Let $f_0, f_1, \cdots, f_{n-1}$ be analytic functions on an open interval L of $\mathbb{R}$. $(f_0, f_1, \cdots, f_{n-1})$ is a *Chebyshev system* on L if any nontrivial linear combination

$$\lambda_0 f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_{n-1} f_{n-1}(x)$$

has at most $n - 1$ isolated zeros on L.

$(f_0, f_1, \cdots, f_{n-1})$ is a *complete Chebyshev system* on L if $(f_0, f_1, \cdots, f_{k-1})$ is a Chebyshev system for all $k = 1, 2, \cdots, n$.

$(f_0, f_1, \cdots, f_{n-1})$ is an *extended complete Chebyshev system* (in short, ECT-system) on L if, for all $k = 1, 2, \cdots, n$, any nontrivial linear combination

$$\lambda_0 f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_{k-1} f_{k-1}(x) \tag{35}$$

has at most $k - 1$ isolated zeros on L counted with multiplicities.

Remark 20. *If $(f_0, f_1, \cdots, f_{n-1})$ is an ECT-system on L, then for each $k = 1, ..., n$, then there exists a linear combination* (35) *with exactly $k - 1$ simple zeros on L (see for instance Remark 3.7 in* [7]*).*

Lemma 21. [6, 11]$(f_0, f_1, \cdots, f_{n-1})$ *is an ECT-system on L if, and only if, for each* $k = 1, 2, \cdots, n$,

$$\Delta_k = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & f_1'(x) & \cdots & f_{k-1}'(x) \\ & & \vdots & \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix} \neq 0.$$

Proposition 22. *There is a real number* $b \in (-4, 0]$ *such that* (J_1, J_2, J_3, J_4) *is an ECT-system in* $(-4, b)$.

Proof. In this proof we use the notation introduced in Lemma 21. Let $f_k(h) = J_{k+1},\ k = 0, 1, 2, 3$. By Lemma 3 and Lemma 5 we have $\Delta_1 = J_1(h) < 0$. It follows from Lemma 8 that the following expansions hold near $h = -4$, with $a_0 = I'(-4) < 0$:

$$\begin{aligned} \Delta_2 &= a_0^2\left(-\frac{1}{1458}(h+4)^2 + \cdots\right), \\ \Delta_3 &= a_0^3\left(\frac{1}{944784}(h+4)^3 + \cdots\right), \\ \Delta_4 &= a_0^4\left(-\frac{1}{6377292} - \frac{47(h+4)}{229582512} + \cdots\right). \end{aligned}$$

Therefore there is a real number $b \in (-4, 0]$ such that $\Delta_i(h) < 0, i = 1, 2, 3, 4$, for $h \in (-4, b)$. The assertion follows from Lemma 21. □

Corollary 23. $J(h) \equiv 0$ *if and only if* $\mu_i = 0,\ i = 1, 2, 3, 4$.

Proof. Suppose that $J(h)$ have the form as (12). If $J(h) \equiv 0$, then $J'(h) = J''(h) = J'''(h) \equiv 0$. In the proof of Proposition (22), we have shown that $\Delta_4 < 0$ as $h \to -4$, which implies that $J(h) = J'(h) = J''(h) = J'''(h) \equiv 0$ if and only if $\mu_i = 0,\ i = 1, 2, 3, 4$. This completes the proof. □

Now we can prove the main results of this paper.

Proof of Theorem 2. By Remark 7, Proposition 15, Proposition 17 and Proposition 19, $J(h)$ has at most three zeros in $(-4, 0)$. Proposition 22 and Remark 20 show that there exist $\mu_i, i = 1, 2, 3, 4$, such that $J(h)$ has exactly three zeros in $(-4, b),\ b \in (-4, 0]$. The theorem follows from Proposition 6. □

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